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# A Rational Approximation to Weierstrass' ℘-Function

By Ulrich Eckhardt

**Abstract.** A rational approximation to Weierstrass' ℘-function in the equianharmonic case for unit period parallelogram is given. With a third-degree numerator polynomial and a fourth-degree denominator polynomial the maximal error for $|z| < 1/\sqrt{3}$ becomes $3 \cdot 10^{-14}$.

The approximation of $\wp(z)$ is then used to calculate a rational approximation to $\wp'(z)$ together with an error bound.

Weierstrass' elliptic functions are frequently used in mathematical physics ([2], [6], [7], [8], [12], [13]). A very interesting new field of application is given in the theory of type-II superconductors [4] and of superfluid helium II [11]. In contrast to other special functions of mathematical physics, however, there are only very few approximations of these functions available. The classical way is to compute them by Laurent series or $\vartheta$-series. Recently, Emersleben [5] gave approximations of generalized zeta functions and Southard [9], [10] published economized polynomial representations.

In this paper, a rational approximation to Weierstrass' ℘-function for the equianharmonic case is presented. This approximation was computed by a method for evaluating lattice sums in solid state physics. In [4], Weierstrass' zeta function was approximated in exactly the same manner.

In the equianharmonic case, the period lattice of Weierstrass' elliptic functions is given in the following way: With the complex numbers

$$(1) \qquad r_j = e^{j\pi i/3}, \quad j = 0, 1, \ldots,$$

the lattice points are defined by

$$(2) \qquad w = w_{mn} = mr_1 + nr_5, \qquad m, n = 0, 1, 2, \ldots.$$

In the more mathematical literature on Weierstrass' elliptic functions, the $r_j$ are premultiplied by a factor 3.059908074 . . . (see [10, 18.13.15]). In view of the physical applications we prefer to have $|r_j| = 1$.

For any complex $z$, Weierstrass' ℘-function is defined as follows:

$$(3) \qquad \wp(z) = \frac{1}{z^2} + \sum_{w \neq 0} \left[ \frac{1}{(z-w)^2} - \frac{1}{w^2} \right].$$

This double sum is unconditionally convergent. We prescribe a fixed succession of summation by renumbering the $w_{mn}$.

Let

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(4)
$$w = w_{n,k,j} = nr_j + kr_{j+2},$$
$$n = 1, 2, 3, \ldots, \quad k = 1, 2, \ldots, n, \quad j = 0, 1, 2, 3, 4, 5.$$

We denote

(5)
$$\sum_{n=\alpha}^{\beta}{}^{(6)} = \sum_{n=\alpha}^{\beta} \sum_{k=1}^{n} \sum_{j=0}^{5},$$

i.e. we are summing over concentric hexagonal shells. This results in the following simpler conditionally convergent series:

(6)
$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}{}^{(6)} \frac{1}{(z-w)^2}.$$

We split this series into three parts

(7a)
$$\wp(z) = \frac{1}{z^2} + 6 \cdot z^4 \cdot \frac{5 + z^6}{(1 - z^6)^2}$$

(7.b)
$$+ \sum_{n=2}^{N}{}^{(6)} \frac{1}{(z-w)^2}$$

(7.c)
$$+ \sum_{n=N+1}^{\infty}{}^{(6)} \frac{1}{(z-w)^2}.$$

(7.b) is a rational function:

(8)
$$\sum_{n=2}^{N}{}^{(6)} \frac{1}{(z-w)^2} = 6 \cdot z^4 \cdot \widetilde{P}(z)/\widetilde{Q}(z);$$

$\widetilde{P}$ and $\widetilde{Q}$ are polynomials whose coefficients can be calculated conveniently and safely (e.g. PL1-FORMAC rational arithmetic).

First, we examine (7.c).

$$\sum_{n=N+1}^{\infty}{}^{(6)} \frac{1}{(z-w)^2} = - \sum_{n=N+1}^{\infty}{}^{(6)} \frac{d}{dz} \frac{1}{z-w}$$

$$= - \sum_{n=N+1}^{\infty} \sum_{k=1}^{n} \frac{d}{dz} \frac{6z^5}{z^6 - (n + kr_2)^6}$$

(9)
$$= - \sum_{n=N+1}^{\infty} \frac{d}{dz} \left[ \frac{6z^5}{n^6} \cdot \frac{1}{\left(1 + \frac{k}{n} r_2\right)^6} \cdot \sum_{\nu=0}^{\alpha-1} \frac{(z/n)^{6\nu}}{\left(1 + \frac{k}{n} r_2\right)^{6\nu}} + \frac{6z^5}{n^6} \cdot \frac{(z/n)^{6\alpha}}{\left(1 + \frac{k}{n} r_2\right)^{6\alpha}} \cdot \frac{1}{\left(1 + \frac{k}{n} r_2\right)^6 - (z/n)^6} \right]$$

$$= \sum_{\nu=0}^{\alpha-1} (36\nu + 30) z^{6\nu+4} \sum_{n=N+1}^{\infty} \frac{1}{n^{6\nu+5}} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\left(1 + \frac{k}{n} r_2\right)^{6\nu+6}} + R_N^{(\alpha)},$$

where

$$R_N^{(\alpha)} = \sum_{n=N+1}^{\infty} \sum_{k=1}^{n} \Bigg[ \frac{1}{n^{6\alpha+6}} \frac{1}{\left(1+\frac{k}{n}r_2\right)^{6\alpha}} \frac{(36\alpha+30)\, z^{6\alpha+4}}{\left(1+\frac{k}{n}r_2\right)^6 - \left(\frac{z}{n}\right)^6} \tag{10}$$
$$+ \frac{1}{n^{6\alpha+12}} \frac{1}{\left(1+\frac{k}{n}r_2\right)^{6\alpha}} \frac{36 z^{6\alpha+10}}{\left[\left(1+\frac{k}{n}r_2\right)^6 - \left(\frac{z}{n}\right)^6\right]^2} \Bigg].$$

For $k = 1, 2, \ldots, n$ is

$$\frac{\sqrt{3}}{2} \leqslant \left| 1 + \frac{k}{n} r_2 \right| \leqslant 1. \tag{11}$$

We confine ourselves to argument values $z$ with

$$|z| \leqslant (N+1)/2 \tag{12}$$

and define

$$\sigma_N^{(k)} := \sum_{n=N+1}^{\infty} \frac{1}{n^k} = \zeta_0(k) - \sum_{n=1}^{N} \frac{1}{n^k}, \tag{13}$$

$\zeta_0(k)$ being the Riemann zeta function. This gives the following estimate for $R_N^{(\alpha)}$:

$$|R_N^{(\alpha)}| \leqslant (36\alpha + 30) \left(\frac{64}{27}\right)^{\alpha} \frac{64}{26}\, \sigma_N^{(6\alpha+5)} |z|^{6\alpha+4} \tag{14}$$
$$+ 36 \left(\frac{64}{27}\right)^{\alpha} \frac{64^2}{26^2}\, \sigma_N^{(6\alpha+11)} |z|^{6\alpha+10}.$$

In (9), the following sum appears:

$$S_n^{(\nu)} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\left(1+\frac{k}{n}r_2\right)^{6\nu}} = \frac{1}{n} \sum_{k=0}^{n}{}' \frac{1}{\left(1+\frac{k}{n}r_2\right)^{6\nu}}. \tag{15}$$

The prime indicates the first and the last term of the sum being multiplied by a factor ½. Denoting by

$$f_\nu(t) = (1 + t \cdot r_2)^{-6\nu}, \tag{16}$$

$S_n^{(\nu)}$ is exactly the trapezoid approximation for the integral

$$I_\nu = \int_0^1 f_\nu(t)\, dt = -\frac{1}{6\nu - 1}, \qquad \nu = 1, 2, \ldots. \tag{17}$$

$S_n^{(\nu)}$ can be expressed by $I_\nu$ by means of the Euler-Maclaurin formula [1, 23.1.30]. In order to do so we denote by $B_{2m}$ the $2m$th Bernoulli number and

$$\Delta_k^{(\nu)} = f_\nu^{(2k-1)}(1) - f_\nu^{(2k-1)}(0), \qquad k = 1, 2, \ldots. \tag{18}$$

Then

$$S_n^{(\nu)} = I_\nu + \sum_{k=1}^{m_\nu - 1} \frac{B_{2k}}{(2k)!} \frac{1}{n^{2k}} \Delta_k^{(\nu)} \tag{19}$$
$$+ \frac{B_{2m_\nu}}{(2m_\nu)!} \frac{1}{n^{2m_\nu}} \frac{1}{n} \sum_{k=0}^{n-1} f_\nu^{(2m_\nu)}\left((1+\theta)\frac{k}{n}\right), \qquad 0 < \theta < 1.$$

Abbreviating

$$c_m = \frac{B_{2m}}{(2m)!}, \qquad \gamma_m^{(\nu)} = \max_{0 \leqslant t \leqslant 1} |f_\nu^{(2m)}(t)|, \tag{20}$$

we get

$$\left| \frac{B_{2m_\nu}}{(2m_\nu)!} \frac{1}{n} \sum_{k=0}^{n-1} f_\nu^{(2m_\nu)}\left((1+\theta)\frac{k}{n}\right) \right| \leqslant c_{m_\nu} \cdot \gamma_{m_\nu}^{(\nu)}. \tag{21}$$

(9), (13) and (19) yield

$$\begin{aligned} \sum_{n=N+1}^{\infty} \frac{1}{(z-w)^2} &= \sum_{\nu=0}^{\alpha-1} (36\nu + 30) z^{6\nu+4} \sum_{k=1}^{m_\nu - 1} \frac{B_{2k}}{(2k)!} \Delta_k^{(\nu+1)} \sigma_N^{(6\nu+2k+5)} \\ &\quad - 6 \sum_{\nu=0}^{\alpha-1} \sigma_N^{(6\nu+5)} z^{6\nu+4} + \rho_{N,m_\nu}^{(\alpha)}. \end{aligned} \tag{22}$$

The error $\rho_{N,m_\nu}^{(\alpha)}$ can be estimated by means of (14), (19) and (21) as follows:

$$\begin{aligned} |\rho_{N,m}^{(\alpha)}| &\leqslant \sum_{\nu=0}^{\alpha-1} (36\nu + 30) c_{m_\nu} \gamma_{m_\nu}^{(\nu)} \sigma_N^{(6\nu+2m_\nu+5)} |z|^{6\nu+4} \\ &\quad + (36\alpha + 30) \left(\frac{64}{27}\right)^\alpha \frac{64}{26} \sigma_N^{(6\alpha+5)} |z|^{6\alpha+4} \\ &\quad + 36 \left(\frac{64}{27}\right)^\alpha \frac{64^2}{26^2} \sigma_N^{(6\alpha+11)} |z|^{6\alpha+10}. \end{aligned} \tag{23}$$

For $\alpha = 1$ we have in particular

$$\begin{aligned} &\sum_{n=N+1}^{\infty}{}^{(6)} \frac{1}{(z-w)^2} \\ &= 30\, z^4 \Big[ -0.2\, \sigma_N^{(5)} - 0.5\, \sigma_N^{(7)} - \frac{14}{15} \sigma_N^{(9)} \\ &\qquad - \sigma_N^{(11)} + 3.3\, \sigma_N^{(13)} + \frac{91}{3} \sigma_N^{(15)} + \frac{2764}{30} \sigma_N^{(17)} + \cdots \Big] \\ &\quad + \rho_{N,m}^{(1)} \end{aligned} \tag{24}$$

and

$$\begin{aligned} |\rho_{N,m}^{(1)}| &\leqslant 30 \cdot c_m \gamma_m^{(1)} \cdot \sigma_N^{(2m+5)} |z|^4 \\ &\quad + 66 \cdot \frac{64^2}{26 \cdot 27} \cdot \sigma_N^{(11)} \cdot |z|^{10} + 36 \cdot \frac{64^3}{27 \cdot 26^2} \sigma_N^{(17)} |z|^{16} \\ &\leqslant 30 \cdot c_m \gamma_m^{(1)} \sigma_N^{(2m+5)} |z|^4 + 385.095 \cdot \sigma_n^{(11)} \cdot |z|^{10} \\ &\quad + 517.050 \cdot \sigma_N^{(17)} |z|^{16}. \end{aligned} \tag{25}$$

Tables for $c_m$, $\gamma_m^{(1)}$ and $\sigma_N^{(k)}$ can be found in [4].

In order to calculate $\wp(z)$ efficiently by the approximation given here, one takes advantage of the fact that this function is doubly periodic, i.e.

$$\wp(z + r_j) = \wp(z). \tag{26}$$

Thus, it is sufficient to know $\wp(z)$ in the unit circle. Using all symmetries of the $\wp$-function in the equianharmonic case, it suffices to know $\wp(z)$ for $|z| \leqslant 1/\sqrt{3}$ (see [10], 18.2.22 to 18.2.23 and 18.13.1 to 18.13.8]). It depends, however, on the computer used, whether this latter reduction will be economical.

Having bounded the argument in this manner, one is able to reduce the polynomials $\widetilde{P}$ and $\widetilde{Q}$, as defined in (8). If we put for abbreviation

$$K_{N,\nu}^{(\alpha)} = (6\nu + 5) \cdot \sum_{k=1}^{m_\nu - 1} \frac{B_{2k}}{(2k)!} \cdot \Delta_k^{(\nu+1)} \cdot \sigma_N^{(6\nu+2k+5)} - \sigma_N^{(6\nu+5)}, \tag{27}$$

we have

$$\wp(z) = \frac{1}{z^2} + 6z^4 \cdot \frac{5 + z^6}{(1 - z^6)^2} + 6z^4 \cdot \frac{\widetilde{P}(z)}{\widetilde{Q}(z)} + 6z^4 \cdot \sum_{\nu=0}^{\alpha-1} K_{N,\nu}^{(\alpha)} \cdot z^{6\nu}. \tag{28}$$

Let

$$\begin{aligned} \hat{P}(z) &= \widetilde{P}(z) + \widetilde{Q}(z) \cdot \sum_{\nu=0}^{\alpha-1} K_{N,\nu}^{(\alpha)} \cdot z^{6\nu} =: \sum \alpha_j z^{6j}, \\ \widetilde{Q}(z) &=: \sum \beta_j z^{6j}. \end{aligned} \tag{29}$$

With increasing $j$ the coefficients $\alpha_j$ and $\beta_j$ become quickly small in modulus. Therefore, we choose integers $k_N$ and $k_D$ and define the truncated polynomials

$$P(z) = \sum_{j=0}^{k_N} \alpha_j z^{6j}, \qquad Q(z) = \sum_{j=0}^{k_D} \beta_j z^{6j}. \tag{30}$$

The truncation error

$$\delta(z) = \hat{P}(z)/\widetilde{Q}(z) - P(z)/Q(z) \tag{31}$$

can easily be estimated. For the approximation

$$\wp(z) \approx \frac{1}{z^2} + 6z^4 \cdot \frac{5 + z^6}{(1 - z^6)^2} + 6z^4 \cdot \frac{P(z)}{Q(z)} \tag{32}$$

we thus have an error of

$$\epsilon(z) = \rho_{N,m}^{(\alpha)} + 6z^4 \cdot \delta(z). \tag{33}$$

To find an approximation to $\wp(z)$ one first has to decide whether one wishes to confine to $|z| \leqslant 1$ or $|z| \leqslant 1/\sqrt{3}$. After that, $N$ is fixed so that the desired accuracy is achieved with a minimum of computer time. In most cases $\alpha = 1$ is preferable. $m$ is chosen in such a way that the error in (19) has nearly the same order of magnitude as $R_N^{(\alpha)}$. Finally, $k_N$ and $k_D$ are chosen such that $6z^4\delta(z)$ has the same order of magnitude as $\rho_{N,m}^{(\alpha)}$.

TABLE I

*The coefficients* $\alpha_j$ *and* $\beta_j$ *of the representation* (32) *for* $N = 20$, $\alpha = 1$, $m = 4$, *and* $j = 0, \ldots, 9$.

| j | $\alpha_j$ | $\beta_j$ |
|---|---|---|
| 0 | - 0.11414 02554 78832 25212 | 1.00000 00000 00000 00000 |
| 1 | 0.01246 21470 34158 86306 | 0.04565 55348 48201 06097 |
| 2 | 0.00007 33190 74452 31537 | - 0.00056 44480 18476 46390 |
| 3 | 0.00000 12310 17921 46078 | - 0.00002 61983 29204 21111 |
| 4 | 0.00000 00026 84924 69697 | 0.00000 02553 31457 29808 |
| 5 | - 0.00000 00000 44046 10396 | 0.00000 00006 22521 91170 |
| 6 | 0.00000 00000 00135 87306 | 0.00000 00000 04512 35409 |
| 7 | 0.00000 00000 00000 21151 | - 0.00000 00000 00000 97714 |
| 8 | 0.00000 00000 00000 00002 | 0.00000 00000 00000 00184 |
| 9 | --- | - 0.00000 00000 00000 00004 |

TABLE II

*The coefficients* $\alpha_j^{(1)}$ *and* $\beta_j^{(1)}$ *of the representation* (37) *for* $N = 20$, $\alpha = 1$, $m = 4$, $j = 0, \ldots, 10$.

| j | $\alpha_j^{(1)}$ | $\beta_j^{(1)}$ |
|---|---|---|
| 0 | - 0.45656 10219 15329 00848 | 1.00000 00000 00000 00000 |
| 1 | 0.13504 37391 64781 42656 | 0.09131 10696 96402 12194 |
| 2 | 0.00293 35592 15775 43401 | 0.00095 55318 25322 37367 |
| 3 | 0.00003 27611 02419 05813 | - 0.00010 39370 10763 52269 |
| 4 | 0.00000 40036 54572 23584 | - 0.00000 15629 32983 73684 |
| 5 | - 0.00000 00446 07821 71793 | 0.00000 00541 34822 32985 |
| 6 | - 0.00000 00004 92565 29248 | 0.00000 00004 63977 63268 |
| 7 | - 0.00000 00000 02669 74929 | - 0.00000 00000 13671 20407 |
| 8 | 0.00000 00000 00007 09648 | 0.00000 00000 00027 39656 |
| 9 | - 0.00000 00000 00000 27512 | 0.00000 00000 00000 08265 |
| 10 | 0.00000 00000 00000 00024 | 0.00000 00000 00000 00274 |

All numbers occurring in the calculations can be computed very accurately. $\sigma_N^{(k)}$ is calculated according to (13) by using Table 23.3 in [1]. This gives a maximum accuracy of about $10^{-20}$. For greater accuracy Table 33 of [3] is preferable.

As an example, an approximation of the $\wp$-function for $N = 20$, $\alpha = 1$ and $m = 4$ was computed. The author is very thankful to Mrs. I. Oellers for excellently written programs. All computations were performed on the IBM 360/67 of the Central Institute for Applied Mathematics of the Nuclear Research Center, Jülich. The coefficients $\alpha_j$ and $\beta_j$ according to (29) and (30) are listed in Table I. For $|z| \leqslant 1$ is $|\rho_{20,4}^{(1)}| \leqslant 3.050 \cdot 10^{-12}$. Choosing $k_N = k_D = 5$, one has

(34) $$|\epsilon(z)| \leqslant 7.678 \cdot 10^{-12}.$$

For $|z| \leqslant 1/\sqrt{3}$, $k_N = 3$, $k_D = 4$ is

(35) $$|\epsilon(z)| \leqslant 3.131 \cdot 10^{-14}.$$

This approximation compares very well with Southard's approximation [9] and [10, 18.13.67].

By differentiating all relevant formulas, one can derive very directly approximations and error estimates for derivatives of Weierstrass' $\wp$-function. We sketch this procedure for the first derivative. Putting (9) into (7) and differentiating, we have as an estimate for the error

$$\left|\frac{d}{dz} R_N^{(\alpha)}\right| + \sum_{\nu=0}^{\alpha-1} (6\nu + 4)(36\nu + 30) \cdot c_{m_\nu}\gamma_{m_\nu}^{(\nu)} \cdot \sigma_N^{(6\nu+2m_\nu+5)} \cdot |z|^{6\nu+3};$$

in particular, for $\alpha = 1$ the error is less than

(36) $$120 c_m \gamma_m^{(1)} \sigma_N^{(2m+5)} |z|^3 + 3.851 \cdot 10^3 \sigma_N^{(11)} |z|^9 + 8.28 \cdot 10^3 \sigma_N^{(17)} |z|^{15}.$$

With $\hat{P}$ and $\widetilde{Q}$ from (29) the corresponding approximation is

(37) $$\begin{aligned} \wp'(z) &\approx -\frac{2}{z^3} + 12z^3 \cdot \frac{10 + 25z^6 + z^{12}}{(1 - z^6)^3} \\ &\quad + 6z^3 \cdot \frac{4 \cdot \hat{P} \cdot \widetilde{Q} + z \cdot \hat{P}' \cdot \widetilde{Q} - z \cdot \hat{P} \cdot \widetilde{Q}'}{\widetilde{Q}^2} \\ &= -\frac{2}{z^3} + 12z^3 \cdot \frac{10 + 25z^6 + z^{12}}{(1 - z^6)^3} + 6z^3 \cdot \frac{P_1(z)}{Q_1(z)}. \end{aligned}$$

The coefficients of $P_1(z) = \Sigma\, \alpha_j^{(1)} \cdot z^{6j}$ and $Q_1(z) = \Sigma \beta_j^{(1)} \cdot z^{6j}$ can be computed easily using the coefficients $\alpha_j$ and $\beta_j$. For $N = 20$, $\alpha = 1$ and $m = 4$ they are given in Table II. The corresponding error estimates are:

For $|z| \leqslant 1$, $k_N = k_D = 6$ is

$$|\epsilon(z)| \leqslant 1.067 \cdot 10^{-10},$$

for $|z| \leqslant 1/\sqrt{3}$, $k_N = k_D = 4$ we have

$$|\epsilon(z)| \leqslant 3.244 \cdot 10^{-13}.$$

FIGURE 1. *The error in approximating* $\wp(z)$ *on the real axis for*

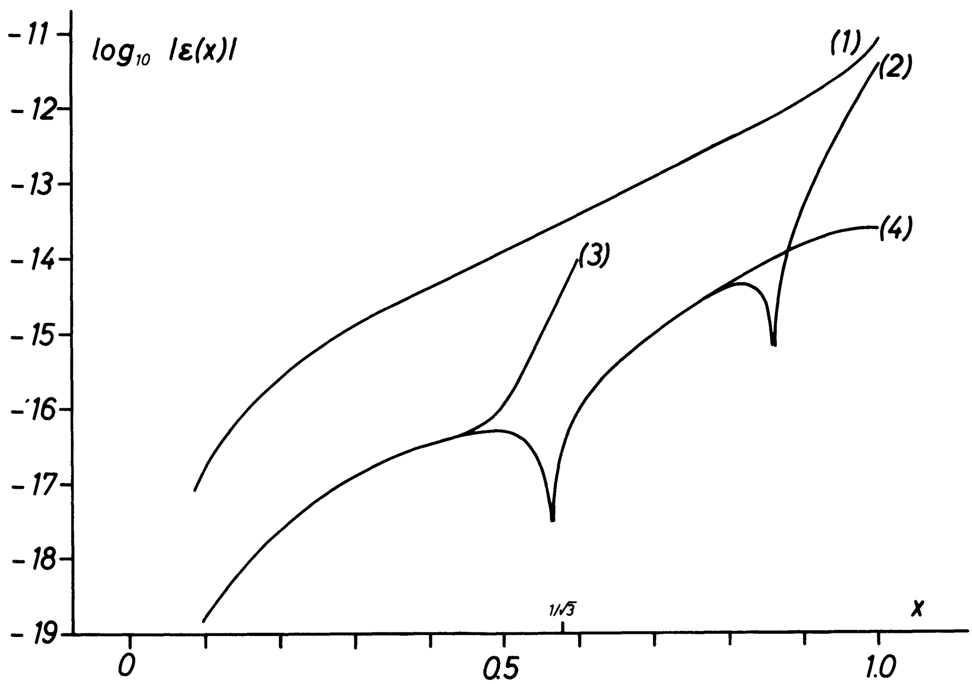
$N = 20, \alpha = 1, m = 4.$

(1) Bound for $\epsilon(x)$ according to (25) and (33) for $k_N = k_D = 5$,

(2) Error for $k_N = k_D = 5$,

(3) Error for $k_N = 3, k_D = 4$; see (35),

(4) Error for $k_N = k_D = 10$.




Nuclear Research Center
Central Institute for Applied Mathematics
D517 Jülich, Germany

1. M. ABRAMOWITZ & I. A. STEGUN, Editors, *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, Nat. Bur. Standards Appl. Math. Series, vol. 55, U.S. Government Printing Office, 1964; reprint, Dover, New York, 1966. MR 34 #8606.

2. N. I. AHIEZER, *Elements of the Theory of Elliptic Functions*, GITTL, Moscow, 1948; 2nd rev. ed., "Nauka", Moscow, 1970. (Russian) MR 12, 409; 44 #5517.

3. H. T. DAVIS, *Tables of the Mathematical Functions*, vol. II, Principia Press of Trinity University, San Antonio, Texas, 1963.

4. U. ECKHARDT, *Zur Berechnung der Weierstrassschen Zeta- und Sigma-Funktion*, Berichte der KFA Jülich, Jül-964-MA, June 1973.

5. O. EMERSLEBEN, "Erweiterung des Konvergenzbereichs einer Potenzreihe durch Herausnahme von Singularitäten, insbesondere zur Berechnung einer Zetafunktion zweiter Ordnung," *Math. Nachr.*, v. 31, 1966, pp. 195–220. MR 33 #4022.

6. E. GRAESER, *Einführung in die Theorie der elliptischen Funktionen und deren Anwendungen*, Verlag von R. Oldenbourg, München, 1950. MR 12, 607.

7. A. G. GREENHILL, *The Application of Elliptic Functions*, Dover, New York, 1959. MR 22 #2724.

8. W. MAGNUS & F. OBERHETTINGER, *Anwendung der elliptischen Funktionen in Physik und Technik,* Die Grundlehren der math. Wissenschaften, Band 55, Springer-Verlag, Berlin, 1949. MR 11, 104.

9. T. H. SOUTHARD, "Approximation and table of the Weierstrass $\wp$ function in the equianharmonic case for real argument," *MTAC*, v. 11, 1957, pp. 99–100. MR 19, 182.

10. T. H. SOUTHARD, "Weierstrass elliptic and related functions," in [1, Chapter 18].

11. V. K. TKAČENKO, "On vortex lattices," *Ž. Èksper. Teoret. Fiz.*, v. 49, 1965, pp. 1875–1883. (Russian)

12. F. TÖLKE, *Praktische Funktionenlehre.* Band II: *Theta-Funktionen und spezielle Weierstrasssche Funktionen*, Springer-Verlag, Berlin and New York, 1966. MR 35 #3089.

13. F. TRICOMI, *Elliptische Funktionen*, Transl. and edited by M. Krafft, Mathematik und ihre Anwendungen in Physik and Technik, Reihe A, Band 20, Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1948. MR 10, 532.